

COMPLEX-VALUED HARMONIC MORPHISMS WITH TOTALLY GEODESIC FIBERS

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ABSTRACT. We study complex-valued harmonic morphisms with totally geodesic fibers. We give a necessary curvature condition for the existence of complex-valued harmonic morphisms with totally geodesic fibers on Einstein manifolds.

1. INTRODUCTION

A harmonic morphism is a map between two Riemannian manifolds that pulls back local harmonic functions to local harmonic functions. The simplest examples of harmonic morphisms are constant maps, real-valued harmonic functions and isometries. A characterization of harmonic morphisms was given by Fuglede and Ishihara, they showed in [4] and [9], respectively, that the harmonic morphisms are exactly the harmonic horizontally weakly conformal maps. If we restrict our attention to the maps where the codomain is a surface then the harmonic morphisms are horizontally weakly conformal maps with minimal fibers at regular points.

Between two surfaces the harmonic morphisms are exactly the weakly conformal maps. Thus locally any harmonic morphism to a surface can be turned into a harmonic morphism to the complex plane by composing with a weakly conformal map.

Local existence of harmonic morphisms can be characterized in terms of foliations. If the codomain is a surface then the existence of a local harmonic morphism is equivalent to the existence of a local conformal foliation with minimal fibers at regular points, see [13].

Baird and Wood found a necessary condition, see [2] Corollary 4.4, on the curvature for local existence of complex-valued harmonic morphisms on three-manifolds. In this case the fibers are geodesics and given any orthonormal basis $\{X, Y\}$ for the horizontal space, the Ricci curvature satisfies

$$\text{Ric}(X, X) = \text{Ric}(Y, Y) \text{ and } \text{Ric}(X, Y) = 0.$$

In three dimensions this is equivalent to

$$\langle R(X, U)U, X \rangle = \langle R(Y, U)U, Y \rangle \text{ and } \langle R(X, U)U, Y \rangle = 0$$

for any vertical unit vector U , which in turn is equivalent to the fact that $K(X_\theta \wedge U)$ is independent of θ where $X_\theta = \cos(\theta)X + \sin(\theta)Y$.

We show in this paper that this is true for any complex-valued submersive harmonic morphism with totally geodesic fibers.

Theorem 1.1. *Let (M, g) and (N^2, h) be a Riemannian manifolds, let $\phi : (M, g) \rightarrow (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers and $p \in M$. Given any $U, V \in \ker(d\phi) = \mathcal{V}_p$ and any orthonormal basis $\{X, Y\}$ for $\mathcal{H}_p = \mathcal{V}_p^\perp$, set $X_\theta = \cos(\theta)X + \sin(\theta)Y$. Then*

$$\langle R(X_\theta \wedge U), X_\theta \wedge V \rangle,$$

is independent of θ .

In four dimensions or more this is stronger than the condition on the Ricci curvature

$$\text{Ric}(X, X) = \text{Ric}(Y, Y) \text{ and } \text{Ric}(X, Y) = 0.$$

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Note that the two examples Example 12.2 and Example 12.3 in [8] do not have totally geodesic fibers, and so are only counter-examples to the Ricci curvature condition in the case of minimal but not totally geodesic fibers.

If we assume that the domain (M, g) is an Einstein manifold, then the curvature operator splits into two blocks and we find that there are at least $\dim(M) - 2$ double eigenvalues for the curvature operator. We use this to give an example of a five dimensional homogeneous Einstein manifold that does not have any submersive harmonic morphism with totally geodesic fibers.

2. THE CURVATURE CONDITION

Let (M, g) and (N, h) be Riemannian manifolds and let $\phi : (M, g) \rightarrow (N, h)$ be a smooth submersion. Let $\mathcal{V} = \ker(d\phi)$ be the vertical distribution and $\mathcal{H} = \mathcal{V}^\perp$ the horizontal distribution associated with ϕ . For two vector fields E, F on M define the fundamental tensors A and B , introduced in [12], by

$$A_E F = \mathcal{V}(\nabla_{\mathcal{H}E} \mathcal{H}F) \text{ and } B_E F = \mathcal{H}(\nabla_{\mathcal{V}E} \mathcal{V}F).$$

The fibers of ϕ are said to be totally geodesic if $B = 0$. The dual A_X^* of A_X satisfies $A_X^* F = -\mathcal{H}(\nabla_X \mathcal{V}F)$ for $X \in \mathcal{H}$ and the dual B_U^* satisfies $B_U^* F = -\mathcal{V}(\nabla_U \mathcal{H}F)$ for $U \in \mathcal{V}$.

In [6] S. Gudmundsson calculated the curvature for a horizontally conformal submersion, we state Proposition 2.1.2, and Theorem 2.2.3 (2) and (3) from this paper. We use the formulation from [3], Lemma 11.1.2 and Theorem 11.2.1 (ii) and (iii).

Proposition 2.1. *Let (M, g) and (N, h) be Riemannian manifolds and let $\phi : (M, g) \rightarrow (N, h)$ be a horizontally conformal submersion with dilation $\lambda : M \rightarrow (0, \infty)$. Let U, V, W be vertical vectors and X, Y be horizontal vectors, then*

$$\begin{aligned} (i) \quad A_X Y &= \frac{1}{2} \mathcal{V}([X, Y]) + \langle X, Y \rangle \mathcal{V}(\text{grad } \ln \lambda) \\ (ii) \quad \langle R(U \wedge V), W \wedge X \rangle &= \langle (\nabla_U B)_V W, X \rangle - \langle (\nabla_V B)_U W, X \rangle \\ (iii) \quad \langle R(U \wedge X, Y \wedge V) \rangle &= \langle (\nabla_U A)_X Y, V \rangle + \langle A_X^* U, A_Y^* V \rangle + \langle (\nabla_X B^*)_U Y, V \rangle \\ &\quad - \langle B_V^* Y, B_U^* X \rangle - 2V(\ln \lambda) \langle A_X Y, U \rangle \end{aligned}$$

It is easy to see that

$$A_X Y + A_Y X = 2\langle X, Y \rangle \mathcal{V}(\text{grad } \ln \lambda) \text{ and } A_X Y - A_Y X = \mathcal{V}([X, Y]).$$

If (N^2, h) is a surface and $\{X, Y\}$ an orthonormal basis for \mathcal{H} and $U \in \mathcal{V}$ then

$$A_X^* U = \langle A_X^* U, X \rangle X + \langle A_X^* U, Y \rangle Y = \langle U, A_X X \rangle X + \langle U, A_X Y \rangle Y.$$

From this we can see that $\langle A_X^* U, A_X^* U \rangle$ does not depend on the direction of X ,

$$\begin{aligned} \langle A_X^* U, A_X^* U \rangle &= \langle U, A_X X \rangle^2 + \langle U, A_X Y \rangle^2 \\ &= \langle U, \mathcal{V}(\text{grad } \ln \lambda) \rangle^2 + \langle U, \frac{1}{2} [X, Y] \rangle^2 \\ &= U(\ln \lambda)^2 + \frac{1}{4} \langle U, [X, Y] \rangle^2. \end{aligned}$$

Since the vertical part of the Lie bracket of horizontal vector fields is a tensor the term $\frac{1}{4} \langle U, [X, Y] \rangle^2$ is in fact independent of our choice of orthonormal basis $\{X, Y\}$ for the horizontal space. To see this, suppose $a^2 + b^2 = 1$, then

$$\begin{aligned} \langle U, [aX + bY, bX - aY] \rangle^2 &= \langle U, -a^2[X, Y] + b^2[Y, X] \rangle^2 \\ &= (-1)^2 \langle U, [X, Y] \rangle^2 \end{aligned}$$

We are now ready to prove Theorem 1.1.

Proof. From Proposition 2.1 (iii) the curvature of a horizontally conformal submersion with totally geodesic fibers is

$$\langle R(U \wedge X, Z \wedge V) = \langle (\nabla_U A)_X Z, V \rangle + \langle A_X^* U, A_Z^* V \rangle - 2V(\ln \lambda) \langle A_X Z, U \rangle$$

for any $U, V \in \mathcal{V}$ and any $X, Z \in \mathcal{H}$. Both sides of the expression are tensors, so we may extend the vectors to vector fields in any way we choose.

$$\begin{aligned} \langle R(X_\theta \wedge U), X_\theta \wedge V \rangle &= \cos^2(\theta) \langle R(X \wedge U), X \wedge V \rangle + \sin^2(\theta) \langle R(Y \wedge U), Y \wedge V \rangle \\ &\quad + \cos(\theta) \sin(\theta) (\langle R(X \wedge U), Y \wedge V \rangle + \langle R(Y \wedge U), X \wedge V \rangle). \end{aligned}$$

Extend X, Y, U, V to unit vector fields, then $2\langle \nabla_U X, X \rangle = U\langle X, X \rangle = 0$ and

$$\begin{aligned} \langle R(X \wedge U), X \wedge V \rangle &= \langle (\nabla_U A)_X X, V \rangle + \langle A_X^* U, A_X^* V \rangle - 2V(\ln \lambda) \langle A_X X, U \rangle \\ &= \langle \nabla_U (A_X X), V \rangle - \langle A_{\nabla_U X} X, V \rangle - \langle A_X (\nabla_U X), V \rangle \\ &\quad + \langle \langle A_X^* U, X \rangle X + \langle A_X^* U, Y \rangle Y, \langle A_X^* V, X \rangle X + \langle A_X^* V, Y \rangle Y \rangle \\ &\quad - 2V(\ln \lambda) \langle \mathcal{V}(\text{grad } \ln \lambda), U \rangle \\ &= \langle \nabla_U \mathcal{V}(\text{grad } \ln \lambda), V \rangle - \langle A_{\nabla_U X} X + A_X (\nabla_U X), V \rangle \\ &\quad + \langle A_X X, U \rangle \langle A_X X, V \rangle + \langle A_X Y, U \rangle \langle A_X Y, V \rangle \\ &\quad - 2V(\ln \lambda) U(\ln \lambda) \\ &= \langle \nabla_U \mathcal{V}(\text{grad } \ln \lambda), V \rangle - \langle \langle \nabla_U X, X \rangle \mathcal{V}(\text{grad } \ln \lambda), V \rangle \\ &\quad + U(\ln \lambda) V(\ln \lambda) + \frac{1}{4} \langle [X, Y], U \rangle \langle [X, Y], V \rangle \\ &\quad - 2U(\ln \lambda) V(\ln \lambda) \\ &= \langle \nabla_U \mathcal{V}(\text{grad } \ln \lambda), V \rangle \\ &\quad + U(\ln \lambda) V(\ln \lambda) + \frac{1}{4} \langle [X, Y], U \rangle \langle [X, Y], V \rangle \\ &\quad - 2U(\ln \lambda) V(\ln \lambda). \end{aligned}$$

A similar calculation gives

$$\begin{aligned} \langle R(Y \wedge U), Y \wedge V \rangle &= \langle \nabla_U \mathcal{V}(\text{grad } \ln \lambda), V \rangle \\ &\quad + U(\ln \lambda) V(\ln \lambda) + \frac{1}{4} \langle [Y, X], U \rangle \langle [Y, X], V \rangle \\ &\quad - 2U(\ln \lambda) V(\ln \lambda), \end{aligned}$$

which is equal to the formula above.

Now since we extended to unit vector fields $\langle \nabla_U X, Y \rangle = -\langle X, \nabla_U Y \rangle$ and so

$$\begin{aligned} \langle R(X \wedge U), Y \wedge V \rangle + \langle R(Y \wedge U), X \wedge V \rangle &= \langle \nabla_U (A_X Y), V \rangle - \langle A_{\nabla_U X} Y, V \rangle - \langle A_X (\nabla_U Y), V \rangle \\ &\quad + \langle \nabla_U (A_Y X), V \rangle - \langle A_{\nabla_U Y} X, V \rangle - \langle A_Y (\nabla_U X), V \rangle \\ &\quad + \langle A_X X, U \rangle \langle A_Y X, V \rangle + \langle A_X Y, U \rangle \langle A_Y Y, V \rangle \\ &\quad + \langle A_Y Y, U \rangle \langle A_X Y, V \rangle + \langle A_Y X, U \rangle \langle A_X X, V \rangle \\ &\quad - 2V(\ln \lambda) \langle A_X Y, U \rangle - 2V(\ln \lambda) \langle A_Y X, U \rangle \\ &= \langle \nabla_U (A_X Y), V \rangle + \langle \nabla_U (A_Y X), V \rangle \\ &\quad - \langle A_{\nabla_U X} Y, V \rangle - \langle A_Y (\nabla_U X), V \rangle \\ &\quad - \langle A_X (\nabla_U Y), V \rangle - \langle A_{\nabla_U Y} X, V \rangle \\ &\quad + \langle \mathcal{V}(\text{grad } \ln \lambda), U \rangle \langle A_Y X + A_X Y, V \rangle \\ &\quad + \langle A_X Y + A_Y X, U \rangle \langle \mathcal{V}(\text{grad } \ln \lambda), V \rangle \\ &\quad - 2V(\ln \lambda) \langle A_X Y + A_Y X, U \rangle \\ &= \langle \nabla_U \mathcal{V}([X, Y]), V \rangle + \langle \nabla_U \mathcal{V}([Y, X]), V \rangle \end{aligned}$$

$$\begin{aligned}
& - \langle \langle X, \nabla_U Y \rangle \mathcal{V}(\text{grad } \ln \lambda), V \rangle \\
& - \langle \langle Y, \nabla_U X \rangle \mathcal{V}(\text{grad } \ln \lambda), V \rangle \\
& = 0.
\end{aligned}$$

So the value of $\langle R(X_\theta \wedge U), X_\theta \wedge V \rangle$ does not depend on θ . \square

3. IMPLICATIONS FOR EINSTEIN MANIFOLDS

Proposition 2.1 (ii), says that for a horizontally conformal submersion with totally geodesic fibers

$$\langle R(U \wedge V), W \wedge X \rangle = 0$$

for all $U, V, W \in \mathcal{V}$ and all $X \in \mathcal{H}$.

Let $\{U_k\}$ be an orthonormal basis for \mathcal{V} and $\{X, Y\}$ an orthonormal basis for \mathcal{H} . If we assume that the domain is an Einstein manifold then

$$\begin{aligned}
0 = \text{Ric}(X, U) &= \langle R(X \wedge Y), Y \wedge U \rangle + \sum_k \langle R(X \wedge U_k), U_k \wedge U \rangle \\
&= \langle R(X \wedge Y), Y \wedge U \rangle
\end{aligned}$$

for all $U \in \mathcal{V}$. This means that the curvature operator R splits into irreducible components

$$\bigwedge^2 T_p M = \left(\bigwedge^2 \mathcal{V} \oplus \bigwedge^2 \mathcal{H} \right) \oplus W,$$

where W is generated by the mixed vectors. So

$$R\left(\bigwedge^2 \mathcal{V} \oplus \bigwedge^2 \mathcal{H}\right) \subseteq \bigwedge^2 \mathcal{V} \oplus \bigwedge^2 \mathcal{H} \text{ and } R(W) \subseteq W.$$

From this the eigenvalues of R are the union of the eigenvalues of $R|_{\bigwedge^2 \mathcal{V} \oplus \bigwedge^2 \mathcal{H}}$ and $R|_W$.

We can define a complex structure J on W by $J(X \wedge U) = Y \wedge U$, and $J(Y \wedge U) = -X \wedge U$. The curvature tensor $R|_W$ is, due to Theorem 1.1, represented by an Hermitian matrix H with respect to this complex structure. Let e_j be an eigenvector to the Hermitian matrix H , then e_j and Je_j represent different real eigenvectors for $R|_W$ with the same eigenvalue, thus $R|_W$ and therefore R has at least $\dim(M) - 2$ double eigenvalues. We get

Proposition 3.1. *Let (M, g) be an Einstein manifold and (N^2, h) be a Riemannian manifold. Let R be the curvature operator of (M, g) at $p \in M$. If there is a submersive harmonic morphism $\phi : (M, g) \rightarrow (N^2, h)$ with totally geodesic fibers then R has at least $\dim(M) - 2$ pairs of eigenvalues.*

In particular, the relationship between the determinants of $R|_W$ and H is $\det(R|_W) = \det(H)^2$. So if F is the characteristic polynomial of H and f the characteristic polynomial of $R|_W$, then $f = F^2$ and F is a factor of $\gcd(f, f')$. Thus $\gcd(f, f')$ is a polynomial of degree at least $\deg(F) = \dim(M) - 2$.

4. EXAMPLES

We give an example of a five dimensional manifold that does not have any conformal foliations with totally geodesic fibers, not even locally. The two homogeneous Einstein manifolds below were found by Alekseevskii in [1], but we use the notation of [11].

Example 4.1. Let S be the five dimensional homogeneous Einstein manifold, see [11] Theorem 1(5). This is a solvable simply connected Lie group corresponding to the Lie algebra \mathfrak{s} given by an orthonormal basis $\{A, X_1, X_2, X_3, X_4\}$ with Lie brackets

$$\begin{aligned}
[X_1, X_2] &= \sqrt{\frac{2}{3}} X_3, [X_1, X_3] = \sqrt{\frac{2}{3}} X_4 \\
[A, X_j] &= \frac{j}{\sqrt{30}} X_j, j = 1, 2, 3, 4.
\end{aligned}$$

A long but strait-forward calculation shows that the curvature operator is given by

$$\frac{1}{30} \begin{bmatrix} 13 & -2\sqrt{5} & -4\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\sqrt{5} & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4\sqrt{5} & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \sqrt{5} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & 0 & 9 & -3\sqrt{5} & 0 & -3\sqrt{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & -3\sqrt{5} & 17 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & -3\sqrt{5} & 5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & 0 & 5 & 0 & 7 \end{bmatrix}$$

with respect to the basis $\{X_1 \wedge X_3, X_2 \wedge A, X_4 \wedge A, X_2 \wedge X_4, X_1 \wedge A, X_3 \wedge A, X_1 \wedge X_2, X_2 \wedge X_3, X_1 \wedge X_4, X_3 \wedge X_4\}$. We find that $\gcd(f(x), f'(x)) = -\frac{4}{15} + x$, which is a polynomial of degree $1 < 3$, so there are no conformal foliations with totally geodesic fibers.

One way to produce foliations on a Lie group G is to find a subalgebra \mathfrak{v} of the Lie algebra \mathfrak{g} of G . The Riemannian metric on G is the left translation of the scalar product on \mathfrak{g} . If \mathfrak{v} corresponds to a closed subgroup K we foliate by left translating this subgroup $\mathcal{F} = \{L_g K\}_{g \in G}$. The foliation has totally geodesic fibers if

$$\langle B(U, V), X \rangle = -\frac{1}{2}(\langle [X, U], V \rangle + \langle [X, V], U \rangle) = 0$$

for all $U, V \in \mathfrak{v}$ and all $X \in \mathfrak{h} = \mathfrak{v}^\perp$, and conformal if

$$(\mathcal{L}_V g)(X, Y) = -\frac{1}{2}(\langle [V, X], Y \rangle + \langle [V, Y], X \rangle) = \nu(V) \langle X, Y \rangle$$

for all $V \in \mathfrak{v}$ and all $X, Y \in \mathfrak{h}$ where ν is a linear functional on \mathfrak{v} .

For the example above. If we define a foliation by left translating $\mathfrak{v} = \{A, X_2, X_4\}$ and $\mathfrak{h} = \{X_1, X_3\}$ we get a foliation with totally geodesic fibers but it is not conformal. If instead we define a foliation by left translating $\mathfrak{v} = \{X_2, X_3, X_4\}$ and $\mathfrak{h} = \{A, X_1\}$ we get a conformal foliation but this does not have totally geodesic fibers, in fact, not even minimal fibers.

Now we give an example of a five dimensional manifold with a conformal foliation with totally geodesic fibers, and see how the curvature operator behaves.

Example 4.2. Let S be the five dimensional homogeneous Einstein manifold, see [11] Theorem 1(4). This is a solvable simply connected Lie group corresponding to the Lie algebra \mathfrak{s} given by an orthonormal basis $\{A, X_1, X_2, X_3, X_4\}$ with Lie brackets

$$\begin{aligned} [X_1, X_2] &= \sqrt{\frac{2}{3}} X_3 \\ [A, X_1] &= \frac{2}{\sqrt{33}} X_1, [A, X_2] = \frac{2}{\sqrt{33}} X_2, [A, X_3] = \frac{4}{\sqrt{33}} X_3, [A, X_4] = \frac{3}{\sqrt{33}} X_4. \end{aligned}$$

If we left translate $\mathfrak{v} = \{A, X_3, X_4\}$ and $\mathfrak{h} = \{X_1, X_2\}$ we get a conformal foliation with totally geodesic fibers. The curvature operator is given by

$$\frac{1}{66} \begin{bmatrix} 41 & -4\sqrt{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4\sqrt{22} & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 2\sqrt{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & -2\sqrt{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\sqrt{22} & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{22} & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

with respect to the basis $\{X_1 \wedge X_2, X_3 \wedge A, X_4 \wedge A, X_3 \wedge X_4, X_1 \wedge A, X_2 \wedge A, X_1 \wedge X_3, X_2 \wedge X_3, X_1 \wedge X_4, X_2 \wedge X_4\}$. We see that the curvature operator satisfies the conclusions of Theorem 1.1.

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